Hydrodynamic stability and the inviscid limit

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It is shown that for appropriately posed problems, the solutions of the linearized Navier–Stokes equations approach those of the linearized Euler equations as the viscosity tends to zero.

1. Introduction

The conventional approach to hydrodynamic stability problems considers the Navier–Stokes equations linearized around some basic stationary flow. Since the viscosity is usually quite small it seems reasonable, as a first approximation, to put it equal to zero at the outset. However, the relation of stability results so obtained to those using the full Navier–Stokes equations has been rather obscure. The difficulty arises for various mathematical reasons.

(1) It is commonly stated (to quote, for instance, Lin 1955, p. 126) '...that there are certain damped solutions in a viscous fluid which, in the limit of vanishing viscosity, do not reduce to solutions of the inviscid equation throughout the whole region of the flow.'

(2) The solutions of the Orr–Sommerfeld equation are quite complicated. In particular it is hard to find asymptotic solutions for small viscosity which are accurate throughout the entire physical region.

(3) The inviscid problem is frequently formulated so that the normal modes obtained are not mathematically complete. (It is readily shown that the Orr–Sommerfeld normal modes *are* complete.) This renders it difficult to see how the inviscid normal modes can be used to solve the appropriate initial-value problem.

Recently (Case 1960) it has been pointed out that there are solutions of the linearized Euler equations of stability theory which are, on occasion, overlooked. This suggests that, under certain conditions which it should be possible to state clearly, results of stability calculations starting from the Navier–Stokes equations should pass in the limit of vanishing kinematic viscosity, i.e. $\nu \to 0$, into the results calculated starting from the Euler equations.

There are several physical arguments which point in this direction. First, it may be noted that in quantum mechanics a rather parallel situation exists. In passing from the Navier–Stokes to the Euler equations the terms of highest order in the derivatives are dropped. Similarly, to pass over to the classical mechanics limit from the Schroedinger equation one must drop the terms of highest order in the derivatives. We do not doubt, though, that in classical mechanics we have a consistent and accurate description of a large, well-defined class of phenomena. A second argument is more important. The Euler equations are readily obtained as a rough approximation from the equations of statistical mechanics. A better approximation gives the Navier–Stokes equations. These in turn are not exact. Indeed, it is possible to derive an improved approximation, which consists of a set of equations which reduce to the Navier–Stokes equations only on omitting certain terms. The essential point is that the omitted terms are those of highest order in the derivatives. Thus, if passing from the Navier–Stokes to the Euler equations completely changes the character of the theory, it would seem that passing from the higher-order equations to the Navier–Stokes equations would do the same. If so, hydrodynamics would be no description of nature at all.

We hope that the main point is clear. It is not that the Euler equations describe as wide a class of phenomena as do the Navier–Stokes equations. (Classical mechanics does not describe as much as does quantum mechanics.) Rather it will be shown that for *appropriately posed and restricted problems* the Euler equations lead to the same results as do the Navier–Stokes equations in the limit of zero viscosity.

Here an approach is followed which leads to the above mentioned 'omitted' solutions. The initial-value problem will be solved with the Laplace transform technique. The advantage is that we gain the additional freedom of an appropriate choice of the inversion contour in the complex plane. In particular it is possible always to stay in regions where it is simple to construct asymptotic (with respect to ν) solutions of the relevant differential equations. It will be shown that passing to the limit of zero viscosity in the solution of the initial-value problem corresponding to the Navier–Stokes equations yields almost exactly the solution of the initial-value problem which follows from the Euler equations. The limitations are that the initial perturbations are the same and are smooth in the sense of physically realizable perturbations. The method of proof also suggests a systematic expansion in terms of inverse powers of the square root of the Reynolds number. However, this possibility is not explored here.

2. Statement of the theorem

For simplicity attention is restricted to two-dimensional parallel flows of a homogeneous, incompressible fluid. The flow is between parallel plates at $y = 0$ and $y = y_1$. In the unperturbed state the flow is described by a component $u_0(y)$ in the x-direction (parallel to the plates) and a y-component $v_0 \equiv 0$. We linearize the Navier–Stokes equations around this basic flow and take a Fourier transform with respect to x and a Laplace transform with respect to time t. If

$$v_{pk}(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{-ikx}\,dx \int_0^{\infty} e^{-pt} v_1(x,y,t)\,dt, \tag{1}$$

where v_1 is the perturbed velocity component in the y-direction, then the resulting equations can be simplified to

$$\nu\left(\frac{d^2}{dy^2}-k^2\right)V_p-(p+iku_0)\,V_p+\nu\left(\frac{d^2}{dy^2}-k^2\right)\frac{iku_0''}{p+iku_0}\,v_{pk} = -V(y,0), \tag{2}$$

with

$$V_p(y) = \left\{\frac{d^2}{dy^2}-k^2-\frac{iku_0''}{p+iku_0}\right\}v_{pk}(y), \tag{3}$$

and
$$V(y,0) = \left(\frac{d^2}{dy^2} - k^2\right) v_k(y,0). \tag{4}$$

These equations are to be solved subject to the boundary conditions
$$v_{pk} = 0 = u_{pk} = \frac{i}{k}\frac{\partial v_{pk}}{\partial y} \tag{5}$$
at $y = 0, y_1$.

In the inviscid limit we start with the Euler equations. The same sequence of approximations and transformations again lead to equation (2)—but with $\nu = 0$. This equation is then to be solved with only the boundary conditions
$$v_{pk}^{inv} = 0 \quad \text{at} \quad y = 0, y_1. \tag{5'}$$

(Here and below the convention is adopted that functions labelled with a superscript *inv* describe solutions of the initial-value problem based on the Euler equations.)

Our essential point is the following theorem:
$$(a) \lim_{\nu \to 0} v_{pk}(y) = v_{pk}^{inv} \quad 0 \leqslant y \leqslant y_1, \tag{6}$$
$$(b) \lim_{\nu \to 0} u_{pk}(y) = \begin{Bmatrix} u_{pk}^{inv} & 0 < y < y_1, \\ 0 & y = 0, y_1. \end{Bmatrix} \tag{7}$$

In view of the introductory remarks this result is hardly surprising. The discontinuous behaviour of u_{pk} at the boundaries is also to be expected. In virtue of equation (5) we have $u_{pk} = 0$ at the boundaries for all ν. This should also be true in the limit. (The discontinuity is only the mathematical description of an infinitesimal boundary layer.)

3. Proof

First note that a formal solution of equation (3) subject to $v_{pk} = 0$ at $y = 0, y_1$ is readily constructed. Let $\phi_{1,2}$ be two solutions of the homogeneous equation
$$\left\{\frac{d^2}{dy^2} - k^2 - \frac{iku_0''}{p + iku_0}\right\}\phi = 0, \tag{8}$$
subject to the conditions
$$\phi_1(0) = \phi_2(y_1) = 0. \tag{9}$$

Then
$$v_{pk}(y) = \int_0^{y_1} G(y, y_0)\, V_p(y_0)\, dy_0, \tag{10}$$
where
$$G(y, y_0) = \frac{\phi_1(y_<)\,\phi_2(y_>)}{W(\phi_1, \phi_2)}. \tag{11}$$

(Here $y_<$ and $y_>$ denote the lesser and greater of y and y_0 respectively.)

Also
$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_2\phi_1' = \text{const.} \quad \text{(independent of } y\text{)}. \tag{12}$$

Explicitly,
$$v_{pk}(y) = \frac{1}{W(\phi_1, \phi_2)}\left\{\phi_2(y)\int_0^y \phi_1(y_0)\, V_p(y_0)\, dy_0 + \phi_1(y)\int_y^{y_1} \phi_2(y_0)\, V_p(y_0)\, dy_0\right\}, \tag{13}$$
and
$$u_{pk}(y) = \frac{i}{kW}\left\{\phi_2'(y)\int_0^y \phi_1(y_0)\, V_p(y_0)\, dy_0 + \phi_1'(y)\int_y^{y_1} \phi_2(y_0)\, V_p(y_0)\, dy_0\right\}. \tag{14}$$

For the inviscid case this provides a complete solution. Thus, with $\nu = 0$ it follows from equation (2) that

$$V_p^{inv}(y) = \frac{V(y,0)}{p+iku_0(y)}. \tag{15}$$

Insertion of this result in equations (13) and (14) yields the transforms of the perturbed flow velocities.

To prove the theorem, consider the solution V_p of equation (2) for very small, but finite, ν. Clearly, it is sufficient† to determine V_p from

$$\left\{\frac{d^2}{dy^2} - ikR\eta(y)\right\} V_p = -RV(y,0), \tag{16}$$

where
$$R = 1/\nu, \tag{17}$$

and
$$\eta(y) = u_0(y) - \frac{i}{k}p. \tag{18}$$

As usual in applying the Laplace transform method the transform variable p is chosen to have a positive real part. This has the result that $\eta(y)$ in (16) never vanishes.

To solve equation (16) we consider the homogeneous equation

$$\left\{\frac{d^2}{dy^2} - ikR\eta(y)\right\} \Lambda = 0. \tag{19}$$

The asymptotic forms of the solutions of this equation are readily obtained by the W.K.B. approximation. There are two solutions $\Lambda_{1,2}$ which *can be chosen* so that for large R (small ν)

$$\Lambda_{1,2}(y) \sim \frac{1}{2^{\frac{1}{2}}(kR)^{\frac{1}{4}}[i\eta(y)]^{\frac{1}{4}}} \exp\left\{\mp (kR)^{\frac{1}{2}} \int_0^y [i\eta(y')]^{\frac{1}{2}}\, dy'\right\}. \tag{20}$$

Since $\eta(y)$ does not vanish, these representations hold for $0 \leqslant y \leqslant y_1$. (There is no Stokes phenomenon.) In equation (20) the multiplicative constants have been chosen so that

$$W(\Lambda_1, \Lambda_2) = 1. \tag{21}$$

The following fact is important. Let

$$\arg \eta(y) = \psi(y). \tag{22}$$

Since $\mathrm{Re}(p) > 0$, we have
$$-\pi \leqslant \psi \leqslant 0. \tag{23}$$

We define the square root in equation (20) so that

$$[i\eta(y)']^{\frac{1}{2}} = |\eta(y')|^{\frac{1}{2}}\{\cos\theta + i\sin\theta\}, \tag{24}$$

where
$$\theta = \tfrac{1}{2}\psi + \tfrac{1}{4}\pi. \tag{25}$$

Then
$$-\tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi. \tag{26}$$

Therefore,
$$\cos\theta(y') > 0 \tag{27}$$

for all y'. From this it follows that, for fixed k and large R, Λ_1 and Λ_2 are, respectively, rapidly decreasing and increasing functions of y.

† This assumes v_{pk} is finite and well behaved in the limit $\nu \to 0$. That this is indeed so is readily verified in the final result.

One solution of equation (16) is

$$V_p^{(0)}(y_0) = -R\int_0^{y_1} \mu(y_0, y_0')\, V(y_0', 0)\, dy_0' \tag{28}$$

with

$$\mu(y_0, y_0') = -\Lambda_1(y_>)\,\Lambda_2(y_<). \tag{29}$$

A general solution is obtained by adding a linear combination of independent solutions of the homogeneous equation [equation (19)]. Thus,

$$V_p(y_0) = V_p^{(0)}(y_0) + E_1\Lambda_1(y_0) + E_2\Lambda_2(y_0). \tag{30}$$

Here E_1 and E_2 are constants which can be determined from the requirements that $u_{pk} = 0$ at $y = 0, y_1$.

From equation (14) we find that these conditions are that

$$\int_0^{y_1} \phi_{1,2} V_p\, dy = 0. \tag{31}$$

Solving for $E_{1,2}$ gives

$$E_1 = \frac{CJ_2 - DI_2}{I_1J_2 - I_2J_1}, \tag{32}$$

$$E_2 = \frac{DI_1 - CJ_1}{I_1J_2 - I_2J_1}, \tag{33}$$

where

$$\begin{pmatrix} C \\ D \end{pmatrix} = -\int_0^{y_1} \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix} V_p^{(0)}\, dy, \tag{34}$$

$$I_{1,2} = \int_0^{y_1} \phi_2 \Lambda_{1,2}\, dy, \tag{35}$$

and

$$J_{1,2} = \int_0^{y_1} \phi_1 \Lambda_{1,2}\, dy. \tag{36}$$

In order to estimate $V_p^{(0)}$ and $E_{1,2}$ in the limit of large R it is useful to invoke the following lemmas.

Let $F(y)$ be a continuous function (independent of R) with bounded first derivative. Let $0 \leqslant \alpha < \beta \leqslant y_1$. Then

$$(1)\quad \int_\alpha^\beta \Lambda_1 F\, dy = \frac{1}{2^{\frac{1}{2}}(kR)^{\frac{3}{4}}}\left\{\exp\left[-(kR)^{\frac{1}{2}}\int_0^\alpha (i\eta)^{\frac{1}{2}}\, dy'\right]\left[\frac{F(\alpha)}{[i\eta(\alpha)]^{\frac{3}{4}}} + O(R^{-\frac{1}{2}})\right]\right\}, \tag{37}$$

$$(2)\quad \int_\alpha^\beta \Lambda_2 F\, dy = \frac{1}{2^{\frac{1}{2}}(kR)^{\frac{3}{4}}}\left\{\exp\left[(kR)^{\frac{1}{2}}\int_0^\beta (i\eta)^{\frac{1}{2}}\, dy'\right]\left[\frac{F(\beta)}{[i\eta(\beta)]^{\frac{3}{4}}} + O(R^{-\frac{1}{2}})\right]\right\}. \tag{38}$$

The proof of these lemmas is based on the rapidly decreasing and increasing properties of Λ_1 and Λ_2 noted above. It uses standard techniques of asymptotic estimation and hence is relegated to the Appendix.

Insertion of these results into (28) yields

$$\begin{aligned} V_p^{(0)}(y_0) &= \frac{V(y_0, 0)}{ik\eta(y_0)} + O(R^{-\frac{1}{2}}) \\ &= \frac{V(y_0, 0)}{p + iku_0(y_0)} + O(R^{-\frac{1}{2}}). \end{aligned} \tag{39}$$

Comparing with equation (15), we see that

$$\lim_{R\to\infty} V_p^{(0)}(y_0) = V_p^{inv}(y_0). \tag{40}$$

The evaluation of $E_{1,2}$ is similar. The results are

$$E_1\Lambda_1(y) \sim \frac{\left\{-\int_0^{y_1} \phi_2 V_p^{inv}\,dy\right\}}{\phi_2(0)} \left\{(kR)^{\frac{1}{2}} \frac{[i\eta(0)]^{\frac{3}{4}}}{[i\eta(y)]^{\frac{1}{4}}}\right\} \exp\left[-(kR)^{\frac{1}{2}}\int_0^{y} [i\eta]^{\frac{1}{2}}\,dy'\right], \tag{41}$$

and $$E_2\Lambda_2(y) \sim \frac{\left\{-\int_0^{y_1} \phi_1 V_p^{inv}\,dy\right\}}{\phi_1(y_1)} \left\{(kR)^{\frac{1}{2}} \frac{[i\eta(y_1)]^{\frac{3}{4}}}{[i\eta(y)]^{\frac{1}{4}}}\right\} \exp\left[-(kR)^{\frac{1}{2}}\int_y^{y_1} [i\eta]^{\frac{1}{2}}\,dy'\right]. \tag{42}$$

Since the real part of $[i\eta(y')]^{\frac{1}{2}}$ is always positive, we see that

$$\lim_{R\to\infty} E_1\Lambda_1(y) = \begin{Bmatrix} 0 & y \neq 0, \\ \infty & y = 0, \end{Bmatrix} \tag{43}$$

and

$$\lim_{R\to\infty} E_2\Lambda_2(y) = \begin{Bmatrix} 0 & y \neq y_1, \\ \infty & y = y_1. \end{Bmatrix} \tag{44}$$

In order to see the nature of the singularity in these limit functions we consider the integral of their product with a smoothly varying function $F(y)$. As in the proof of the lemmas (Appendix A) one finds that

$$\lim_{R\to\infty}\int_0^{y_1} F(y)\,E_1\Lambda_1(y)\,dy = -\left(\frac{\int_0^{y_1}\phi_2 V_p^{inv}\,dy}{\phi_2(0)}\right)F(0), \tag{45}$$

and

$$\lim_{R\to\infty}\int_0^{y_1} F(y)\,E_2\Lambda_2(y)\,dy = -\left(\frac{\int_0^{y_1}\phi_1 V_p^{inv}\,dy}{\phi_1(y_1)}\right)F(y_1). \tag{46}$$

These results can be stated concisely in the form

$$\lim_{R\to\infty} E_1\Lambda_1(y) = -\left(\frac{\int_0^{y_1}\phi_2 V_p^{inv}\,dy}{\phi_2(0)}\right)2\delta(y) \tag{47}$$

and

$$\lim_{R\to\infty} E_2\Lambda_2(y) = -\left(\frac{\int_0^{y_1}\phi_1 V_p^{inv}\,dy}{\phi_1(y_1)}\right)2\delta(y_1-y). \tag{48}$$

(Here the delta function is interpreted so that

$$\int_0^{y_1}\delta(y)\,dy = \tfrac{1}{2}.) \tag{49}$$

Thus, we have found that

$$\lim_{R\to\infty} V_p(y_0) = V_p^{inv} - \left(\frac{\int_0^{y_1}\phi_2 V_p^{inv}\,dy}{\phi_2(0)}\right)2\delta(y_0) - \left(\frac{\int_0^{y_1}\phi_1 V_p^{inv}\,dy}{\phi_1(y_1)}\right)2\delta(y_1-y_0). \tag{50}$$

If this form is inserted into equations (13) and (14), the theorem follows. [The difference in behaviour between v_{pk} and u_{pk} at the boundary points occurs because $\phi_1(0) = 0 = \phi_2(y_1)$ which is not true of $\phi_1'(0)$ and $\phi_2'(y_1)$.]

4. Conclusion

In the previous section an identity between the Laplace transform of the inviscid solution and the limit for small viscosity of the transform of the viscous solution was demonstrated. We would like to conclude from this that the solutions of the initial-value problem coincide in the limit. To do this we must show that the limiting process and the Laplace inversion integration can be interchanged. It is readily seen that the only difficulty that could occur is that the region of integration over p is infinite. However, for large p we may neglect the terms involving $u_0(y)$ in the basic equations (i.e. for large p we can regard the unperturbed flow as quiescent). Thus, to conclude that the solutions of the initial-value problems are identical it is sufficient to show this for the problem with $u_0(y) \equiv 0$. This is done in Appendix B. (The results there also show the form of the functions for finite viscosity which pass over into the singular functions described above.)

The conclusion is then the following. Consider a given initial perturbation. The flow at a fixed time later is, except for the singular behaviour of the tangential velocity at the boundaries, exactly the same whether computed with the Euler equations or with the Navier–Stokes equations and passage to the limit of zero viscosity.

The restriction to a fixed (finite) value of the time is important. We cannot justify interchange of the limits $\nu \to 0$ and $t \to \infty$. Thus, in the example in Appendix B, it is seen that

$$\lim_{t \to \infty} u_1, v_1 = 0 \quad \text{(all finite } \nu\text{)}, \tag{51}$$

but

$$\lim_{t \to \infty} u_1{}^{inv}, v^{inv} \neq 0. \tag{52}$$

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Appendix A: proof of the lemmas

Since the proofs are quite similar we restrict ourselves to sketching a proof for the first lemma.

It is desired to show that if $F(y)$ is a slowly varying function then, for large R,

$$\begin{aligned} K &\equiv \int_\alpha^\beta \Lambda_1(y)\, F(y)\, dy \\ &= \frac{1}{2^{\frac{1}{2}}(kR)^{\frac{3}{4}}}\left\{\exp\left[-(kR)^{\frac{1}{2}}\int_0^\alpha (i\eta)^{\frac{1}{2}}\, dy\right]\left[\frac{F(\alpha)}{[i\eta(\alpha)]^{\frac{3}{4}}} + O(R^{-\frac{1}{2}})\right]\right\}. \end{aligned} \tag{A 1}$$

For large R we use the asymptotic representation, equation (20), for Λ_1. Then

$$K \sim \frac{1}{2^{\frac{1}{2}}(kR)^{\frac{1}{4}}}\int_{\alpha}^{\beta}\frac{F(y)}{[i\eta(y)]^{\frac{1}{4}}}\exp\left[-(kR)^{\frac{1}{2}}\int_0^y (i\eta)^{\frac{1}{2}}\,dy'\right]dy. \tag{A 2}$$

Insert the identity

$$\exp\left[-(kR)^{\frac{1}{2}}\int_0^y (i\eta)^{\frac{1}{2}}\,dy'\right] = -\frac{1}{(kR)^{\frac{1}{2}}}\frac{1}{[i\eta(y)]^{\frac{1}{2}}}\frac{d}{dy}\exp\left[-(kR)^{\frac{1}{2}}\int_0^y (i\eta)\,dy'\right], \tag{A 3}$$

and integrate by parts. Since the exponential factor vanishes rapidly with increasing y it is clear that the main contribution is that due to the part evaluated at the lower limit. If $F(y)$ is subject to the appropriate restrictions, we see by repeating the process in the remaining integral that the corrections are of the indicated order of magnitude. In this way the result, equation (A 1), is found for K.

Appendix B

It is necessary, as indicated in § 4, to show the identity of the solutions of the initial-value problem for the special case $u_0(y) = 0$. To keep the algebra simple we consider $y_1 = \infty$ and

$$V(y, 0) = \delta(y - y_0'). \tag{B 1}$$

Thus, for finite viscosity, we are to solve the equations

$$V_p = \left\{\frac{d^2}{dy^2} - k^2\right\} v_{pk}, \tag{B 2}$$

and

$$\nu\left(\frac{d^2}{dy^2} - k^2\right) V_p - pV_p = -\delta(y - y_0'), \tag{B 3}$$

subject to the boundary conditions

$$v_{pk} = 0 = u_{pk} = \frac{i}{k}\frac{\partial v_{pk}}{\partial y} \quad \text{at} \quad y = 0, \infty. \tag{B 4}$$

For the inviscid problem equations (B 2) and (B 3) with $\nu = 0$ are to be solved using only the condition

$$v_{pk}^{inv} = 0 \quad \text{at} \quad y = 0, \infty. \tag{B 5}$$

The solution of equation (B 2) which satisfies equation (B 5) is, in either case,

$$v_{pk}(y) = \int_0^\infty G(y, y_0)\, V(y_0)\, dy_0, \tag{B 6}$$

where (assuming $k > 0$)

$$G(y, y_0) = -\frac{1}{k} e^{-ky_>} \sinh ky_<. \tag{B 7}$$

In the inviscid case we find from equation (B 3) that

$$V_p^{inv} = \frac{\delta(y - y_0')}{p}. \tag{B 8}$$

Therefore

$$v_{pk}^{inv}(y) = G(y, y_0')/p. \tag{B 9}$$

Since the inverse Laplace transform of $1/p$ is 1 for all $t > 0$ we see that

$$v_1^{inv}(y, t) = G(y, y_0') \quad \text{for} \quad t > 0. \tag{B 10}$$

For finite ν the general solution of equation (B 3) subject to the condition of vanishing at infinity is

$$V_p(y_0) = V_p^{(0)}(y_0) + E_1 \Lambda_1(y_0), \tag{B 11}$$

where

$$V_p^{(0)}(y_0) = -R\mu(y_0, y_0'), \tag{B 12}$$

$$\mu(y_0, y_0') = -\Lambda_1(y_>)\Lambda_2(y_<), \tag{B 13}$$

and

$$\Lambda_{1,2} = \frac{e^{\mp\gamma y}}{(2\gamma)^{\frac{1}{2}}}, \tag{B 14}$$

with

$$\gamma = R^{\frac{1}{2}}(p + k^2/R)^{\frac{1}{2}}. \tag{B 15}$$

The constant E_1 is determined by the requirement

$$\frac{\partial v_{pk}}{\partial y} = 0 \quad \text{at } y = 0.$$

This gives

$$E_1 \Lambda_1(y_0) = \left\{ -\frac{\int_0^\infty V_p^{(0)} e^{-ky}\, dy}{\int_0^\infty \Lambda_1 e^{-ky}\, dy} \right\} \Lambda_1(y_0)$$

$$= -\frac{R}{\gamma}\, e^{-ky_0} e^{-\gamma y_0} + \frac{R}{2\gamma} e^{-\gamma(y_0 + y_0')} - R\, e^{-\gamma y_0} \frac{[e^{-ky} - e^{-\gamma y_0'}]}{\gamma - k}. \tag{B 16}$$

The inverse Laplace transformation then gives

$$v_1(y, t) = \int_0^\infty G(y, y_0) \{ V^{(0)}(y_0, t) + (E_1 \Lambda_1)(y_0, t) \}\, dy_0, \tag{B 17}$$

with

$$V^{(0)}(y_0, t) = \frac{1}{2\pi i} \int_C V_p^{(0)}(y_0)\, e^{pt} dp, \tag{B 18}$$

$$(E_1 \Lambda_1)(y_0, t) = \frac{1}{2\pi i} \int_C E_1 \Lambda_1 e^{pt} dp. \tag{B 19}$$

Here C is a contour parallel to the imaginary axis and to the right of all singularities of the integrand. After some straightforward manipulation these become

$$V^{(0)}(y_0, t) = \Delta(y_0 - y_0', t), \tag{B 20}$$

$$(E_1 \Lambda_1)(y_0, t) = -2e^{-ky_0'} \Delta(y_0, t) + \Delta(y_0 + y_0', t) + \Gamma(y_0, t), \tag{B 21}$$

where

$$\Delta(y, t) = \frac{e^{-k^2 t/R}}{2\pi} \int_{-\infty}^{\infty} \{\exp[isy - s^2 t/R]\}\, ds, \tag{B 22}$$

and

$$\Gamma(y, t) = \frac{ik}{\pi} \int_{-\infty}^{\infty} \frac{e^{isy}[e^{-ky_0'} - e^{isy_0'}]\, e^{-s^2 t/R}\, ds}{s - ik}. \tag{B 23}$$

Passing to the limit in the integral expression for Δ clearly gives

$$\lim_{R \to \infty} \Delta(y, t) = \delta(y). \tag{B 24}$$

We can, of course, evaluate $\Delta(y, t)$ for all finite ν also. This gives

$$\Delta(y, t) = e^{-k^2 t/R} \frac{\{e^{-y^2 R/4t}\}}{(4\pi t/R)}. \tag{B 25}$$

Except for the first factor this is just the familiar heat pole solution. It vanishes as $R \to \infty$ for all fixed $y \neq 0$ and is such that the integral over all y is $e^{-k^2t/R}$. (This serves again to justify equation (B 24).) Inserting the expression for $V^{(0)}$ in terms of Δ into equation (B 17) shows that for small, but finite viscosity, the first term is just the inviscid result averaged with respect to y_0' over a distance which is of order $(4t/R)^{\frac{1}{2}}$.

Since the function Γ is non-singular when $R \to \infty$, it suffices to pass to the limit under the integration sign. Thus

$$\lim_{R\to\infty} \Gamma(y_0, t) = \frac{ik}{\pi}\int_{-\infty}^{\infty} e^{isy_0}\frac{\{e^{-ky_0'} - e^{isy_0'}\}}{s-ik}\,ds. \tag{B 26}$$

Closing the contour with a large circle in the upper half plane yields then

$$\lim_{R\to\infty} \Gamma(y, t) = 0. \tag{B 27}$$

Combining these results shows that

$$\lim_{R\to\infty} \{V^{(0)}(y_0, t) + (E_1\Lambda_1)(y_0, t)\} = \delta(y_0 - y_0') - 2e^{-ky'_0}\delta(y_0). \tag{B 28}$$

Using this result in (B 17) then gives

$$\lim_{R\to\infty} v_1(y, t) = v_1^{inv}(y, t) \qquad 0 \leqslant y < \infty, \tag{B 29}$$

$$\lim_{R\to\infty} u_1(y, t) = \frac{i}{k}\frac{\partial v_1}{\partial y}(y, t) = \begin{cases} u_1(y, t) & 0 < y < \infty, \\ 0 & y = 0. \end{cases} \tag{B 30}$$